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CONTRACTIBLE EDGES IN TRIANGLE-FREE GRAPHS

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An edge of a graph is called *k-contractible* if the contraction of the edge results in a k-connected graph. Thomassen [5] proved that every k-connected graph of girth at least four has a k-contractible edge. In this paper, we study the distribution of k-contractible edges in triangle-free graphs and show the following: When $k\geqq 2$, every k-connected graph of girth at least four and order $n\geqq 3k$, has $n+(3/2)k^2-3k$ or more k-contractible edges.

In this paper, we consider only finite simple graphs. An edge of a graph is called *k-contractible* if the contraction of the edge results in a k-connected graph. In [4], Thomassen proved that every 3-connected graph of order at least five has a 3-contractible edge, and in [1], Ando, Enomoto and Saito proved that every 3-connected graph of order $n\geqq 5$ has at least $n/2$ 3-contractible edges.

The similar conclusion does not hold for k-connected graphs when $k\geqq 4$. Thomassen [5] remarked that for $k\geqq 4$ there exist infinitely many k-connected graphs in which no edge is k-contractible. However, for graphs of girth at least four, the existence of k-contractible edges in k-connected graphs is assured by Thomassen.

Theorem A (Thomassen [5]). *Every k-connected graph of girth at least four has a k-contractible edge.* ∎

Thomassen and Toft [6] obtained a stronger result concerning 3-connected graphs of girth at least four.

Theorem B (Thomassen and Toft [6]. *Every 3-connected graph of girth at least four has a cycle consisting of 3-contractible edges.* ∎

The purpose of this paper is to extend Theorem A and Theorem B.

Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively. If the end-vertices of an edge e are x and y, we write $e=xy$. For $x\in V(G)$, $\Gamma_G(x)$ is the set of vertices adjacent to x and $d_G(x)$ is the degree of x in G. For $S\subset V(G)$, $\langle S\rangle_G$ is the subgraph of G induced by S and $G-S=\langle V(G)-S\rangle_G$. A set S is called a *k-cutset* if $G-S$ is disconnected and $|S|=k$. We denote by $w(G)$ the number of components of G and by $|G|$ the order of G. A subset S of $V(G)$ is called independent if $\langle S\rangle_G$ is totally disconnected. The notations not explained here are found in [2].

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Let $\mathscr{C}_k(G)$ be the set of k-cutsets which are not independent.

$$\mathscr{C}_k(G) = \{S \subset V(G) \,|\, |S| = k,\ G-S \text{ is disconnected and } \langle S\rangle_G \neq \overline{K}_k\}$$

and let

$$\mathscr{C}_k(x;\ G) = \{S \in \mathscr{C}_k(G) \,|\, x \in S\}.$$

Let

$$N_G^{(k)}(x) = \{y \in \Gamma_G(x) \,|\, \text{The edge } xy \text{ is } k\text{-contractible.}\}$$

and

$$M_G^{(k)}(x) = \{y \in \Gamma_G(x) \,|\, \text{The edge } xy \text{ is not } k\text{-contractible.}\}$$

First, we remark that if $S \in \mathscr{C}_k(G)$, each component of $G-S$ is large.

Proposition 1. *Let G be a k-connected graph of girth at least four, $S \in \mathscr{C}_k(G)$ and A be a component of $G-S$. Then $|A| \geqq k$.*

Proof. First assume $|A|=1$, say $A=\{a\}$. Then $\Gamma_G(a)=S$. Since S is not independent, G has a 3-cycle, contradicting the assumption on the girth of G. Hence $|A| \geqq 2$ and A has a pair of adjacent vertices of G, say a and b. Since G has no 3-cycle, $\Gamma_G(a) \cap \Gamma_G(b) = \emptyset$. It is obvious that $\Gamma_G(a) \cup \Gamma_G(b) \subset A \cup S$. Therefore,

$$|A|+|S| \geqq |\Gamma_G(a) \cup \Gamma_G(b)| = |\Gamma_G(a)|+|\Gamma_G(b)| \geqq 2k.$$

Since $|S|=k$, we have $|A| \geqq k$. ∎

Theorem 2. *Let G be a k-connected graph of girth at least four. Let $x \in V(G)$ and $S \in \mathscr{C}_k(x; G)$. Then $A \cap N_G^{(k)}(x) \neq \emptyset$ for any component A of $G-S$.*

Proof. Assume $A \cap N_G^{(k)}(x) = \emptyset$ for some $S \in \mathscr{C}_k(x; G)$ and some component A of $G-S$. Take S and A such that $|A|$ is least possible. Let $B = V(G)-(A \cup S)$. Since G is k-connected and $S \in \mathscr{C}_k(x; G)$, $\Gamma_G(x) \cap A \neq \varphi$. By the assumption, $M_G^{(k)}(x) \cap A \neq \emptyset$, say $y \in M_G^{(k)}(x) \cap A$. Then there exists $T \in \mathscr{C}_k(x; G)$ such that $y \in T$. Take a component C of $G-T$ and let $D = V(G)-(C \cup T)$. Let

$$U_1 = (S \cap C) \cup (S \cap T) \cup (A \cap T)$$

and

$$U_2 = (S \cap D) \cup (S \cap T) \cup (B \cap T).$$

Assume $A \cap C \neq \emptyset$. Since U_1 is a cutset of G, $|U_1| \geqq k$. If $|U_1|=k$, then $U_1 \in \mathscr{C}_k(x; G)$. This contradicts the minimality of A since $A \cap C \subsetneqq A$. Hence $|U_1| \geqq k+1$. Since $|U_1|+|U_2| = |S|+|T| = 2k$, $|U_2| \leqq k-1$ and hence $B \cap D = \emptyset$. On the other hand, $|D| \geqq k$ by Proposition 1. Since $|S \cap D| \leqq k-1$, $A \cap D \neq \emptyset$. By the same argument as above, we have $B \cap C = \emptyset$. Therefore, $B \subset T$. Since $x, y \in T-B$, $|B| \leqq k-2$. This contradicts Proposition 1. Hence we have $A \cap C = \emptyset$.

Similarly, we can deduce that $A \cap D = \emptyset$. Hence $A \subset T$ and $|A| \leqq k-1$. This is a contradiction. Therefore, the conclusion follows. ∎

An extension of Theorem B is obtained easily from Theorem 2.

Theorem 3. *Let G be a k-connected graph of girth at least four, where $k \geqq 2$, and $x \in V(G)$. Then $|N_G^{(k)}(x)| \geqq 2$.*

Proof. If $M_G^{(k)}(x)=\emptyset$, then $|N_G^{(k)}(x)|=|\Gamma_G(x)|\geqq k\geqq 2$. If $M_G^{(k)}(x)\neq\emptyset$, say $y\in M_G^{(k)}(x)$, then there exists $S\in\mathscr{C}_k(x;G)$ such that $y\in S$. By Theorem 2, $|N_G^{(k)}(x)|\geqq w(G-S)\geqq 2$. ▮

For $X\subseteqq V(G)$, define $\mathscr{C}_X(G)$ by

$$\mathscr{C}_X(G)=\{S\in\mathscr{C}_k(G)\mid G-S \text{ has a component which is disjoint from } X\}.$$

For $S\in\mathscr{C}_X(G)$, let $A_X(S)$ be a component of $G-S$ of least cardinality which is disjoint from X.

Theorem 4. *Let G be a k-connected graph of girth at least four and $X\subset V(G)$. Suppose $\mathscr{C}_X(G)\neq\emptyset$. Take $S\in\mathscr{C}_X(G)$ such that $|A_X(S)|$ is least possible. Then for $x\in A_X(S)$, $M_G^{(k)}(x)=\emptyset$.*

Proof. Let $B=V(G)-(S\cup A_X(S))$. Assume $M_G^{(k)}(x)\neq\emptyset$, say $y\in M_G^{(k)}(x)$. Then there exists $T\in\mathscr{C}_k(x;G)$ such that $y\in T$. Let C be one of the components of $G-T$ and let $D=V(G)-(C\cup T)$. By Proposition 1, $|A_X(S)|\geqq k$, $|B|\geqq k$, $|C|\geqq k$ and $|D|\geqq k$. Let

$$U_1=(S\cap C)\cup(S\cap T)\cup(A_X(S)\cap T)$$

and

$$U_2=(S\cap D)\cup(S\cap T)\cup(B\cap T).$$

First we claim that $A_X(S)\cap C=\emptyset$. Assume $A_X(S)\cap C\neq\emptyset$. Then U_1 is a cutset of G and $|U_1|\geqq k$. If $|U_1|=k$, then $U_1\in\mathscr{C}_X(G)$ since $A_X(S)\cap C\cap X\subseteqq A_X(S)\cap X=\emptyset$. This contradicts the minimality of $A_X(S)$ since $A_X(S)\cap C\subsetneqq A_X(S)$. Hence we have $|U_1|\geqq k+1$. Since $|U_1|+|U_2|=|S|+|T|=2k$, $|U_2|\leqq k-1$. This implies $B\cap D=\emptyset$ since G is k-connected. Since $|B|\geqq k$ and $|B\cap T|\leqq k-2$, we have $B\cap C\neq\emptyset$. Let

$$U_3=(S\cap C)\cup(S\cap T)\cup(B\cap T)$$

and

$$U_4=(A_X(S)\cap T)\cup(S\cap T)\cup(S\cap D).$$

Then $|U_3|\geqq k$ and $|U_4|\leqq k$. If $A_X(S)\cap D\neq\emptyset$, then $|U_4|=k$ and $U_4\in\mathscr{C}_X(G)$, which contradicts the minimality of $A_X(S)$. Hence $A_X(S)\cap D=\emptyset$ and $D\subset S$. However, $|D|\geqq k$ and $|S|=k$. So we have $D=S$. This implies $|U_3|=|B\cap T|\leqq k-2$, which is a contradiction. Hence the claim that $A_X(S)\cap C=\emptyset$ follows.

Similarly, we can prove that $A_X(S)\cap D=\emptyset$. These imply that $A_X(S)\subseteqq T$. Since $|A_X(S)|\geqq k$ and $|T|=k$, $T=A_X(S)$ and $S\cap T=B\cap T=\emptyset$. Hence $B\cap C\neq\emptyset$ or $B\cap D\neq\emptyset$. Without loss of generality, we may assume $B\cap D\neq\emptyset$. Then $|U_2|\geqq k$, which implies $S=U_2$ and that $S\cap C=B\cap C=\emptyset$. Now we have $B\cap C=S\cap C=A_X(S)\cap C=\emptyset$. This is a contradiction and the theorem follows. ▮

Suppose $\mathscr{C}_\Phi(G)=\mathscr{C}(G)\neq\emptyset$. Take $S_0\in\mathscr{C}(G)$ such that $|A_\Phi(S_0)|$ is least possible and let $A_0=A_\Phi(S_0)$. Let $X_0=A_0\cup S_0$. Then $S_0\in\mathscr{C}_{X_0}(G)$. Take $S_1\in\mathscr{C}_{X_0}(G)$ such that $|A_{X_0}(S_1)|$ is least possible and let $A_1=A_{X_0}(S_1)$. Then, by Theorem 4,

Corollary 5. $M_G^{(k)}(x)=\emptyset$ *for any* $x\in A_0\cup A_1$. ▮

Also by Theorem 2, the following lemma can be seen easily.

Lemma 6. *Let G be k-connected graph of girth at least four. Suppose $\mathscr{C}(G)\neq\emptyset$ and define A_0 and A_1 as above. For $x\in S_i$, $N_G^{(k)}(x)-(A_i\cup S_i)\neq\emptyset$ $(i=0, 1)$.* ∎

In the proof of our next lemma, we use the following well-known proposition.

Proposition C (See [3] Theorem 6 of Chapter IV). *Let G be a graph of girth at least four. Then $|E(G)|\leqq(1/4)|G|^2$.* ∎

Lemma 7. *Let G be a k-connected graph of girth at least four, $S\in\mathscr{C}(G)$ and A be one of the components of $G-S$. Suppose $k\geqq 2$. Then*

$$e_G(S, A)+e_G(A)\geqq\frac{3}{4}k^2-k+|A|.$$

Proof. Let $e_G(S, A)=s$, $e_G(A)=t$ and $|A|=a$. Since the minimum degree of G is not less than k,

$$(1)\qquad s+2t=\sum_{v\in A}d_G(v)\geqq ka.$$

By Proposition C,

$$(2)\qquad t\leqq\frac{1}{4}a^2.$$

Since $S\in\mathscr{C}(G)$, $\Gamma_G(x)\cap S\neq\emptyset$ for any $x\in S$. Hence

$$(3)\qquad s\geqq k.$$

By Proposition 1,

$$(4)\qquad a\geqq k.$$

First consider the case when $a\leqq 3k-4$. Then $s+t=s+2t-t\geqq ka-\left(\frac{1}{4}\right)a^2$ by (1) and (2). On the other hand,

$$\left(ka-\frac{1}{4}a^2\right)-\left(\frac{3}{4}k^2-k+a\right)=\frac{1}{4}(a-k)(3k-4-a)\geqq 0$$

by (4). Hence $s+t\geqq\left(\frac{3}{4}\right)k^2-k+a$. Next, consider the case $a\geqq 3k-3$. Then $s+t\geqq$ $\geqq\left(\frac{1}{2}\right)(s+s+2t)\geqq\left(\frac{1}{2}\right)(k+ka)$ by (1) and (3). Moreover,

$$k+ka-2\left(\frac{3}{4}k^2-k+a\right)=\frac{1}{2}(k-2)(2a-3k)\geqq\frac{1}{2}(k-2)\{2(3k-3)-3k\}\quad(k\geqq 2)$$

$$=\frac{3}{2}(k-2)^2\geqq 0.$$

Therefore, also in this case we have $s+t\geqq\left(\frac{3}{4}\right)k^2-k+a$. ∎

Now, we can obtain the lower bound of the number of k-contractible edges in a k-connected, triangle-free graph. Let $E^{(k)}(G)$ be the set of k-contractible edges in G.

Theorem 8. *Let G be a k-connected graph of girth at least four and suppose $k \geqq 2$ and $|G| \geqq 3k$. Then $|E^{(k)}(G)| \geqq |G| + \frac{3}{2}k^2 - 3k$.*

Proof. If $\mathscr{C}(G) \neq \emptyset$, we can define A_0, A_1, S_0 and S_1 as in the paragraph preceding Corollary 5. Let $T_0 = S_0 - S_1$, $T_1 = S_1 - S_0$, $U_0 = S_0 \cap S_1$ and $U_1 = V(G) - (S_0 \cup S_1 \cup \cup A_0 \cup A_1)$. Let $|A_i| = a_i$ $(i = 0, 1)$ and $b = |U_0|$. Since $\Gamma_G(x) \cap A_1 \neq \emptyset$ for any $x \in S_1$, $S_1 \cap A_0 = \emptyset$. Hence by Theorem 3, Corollary 5 and Lemma 6,

$$2|E^{(k)}(G)| \geqq 2|U_1| + 2\{e_G(A_0) + e_G(A_0, S_0) + e_G(A_1) + e(A_1, S_1)\} + |T_0| + |T_1| =$$
$$= 2\{|G| - (a_0 + a_1 + 2k - b)\} + 2\{e_G(A_0) + e_G(A_0, S_0) + e_G(A_1) + e_G(A_1, S_1)\}$$
$$+ 2(k - b).$$

Since $e_G(A_i) + e_G(A_i, S_i) \geqq \left(\frac{3}{4}\right)k^2 - k + a_i$ $(i = 0, 1)$ by Lemma 7,

$$2|E^{(k)}(G)| \geqq 2\{|G| - (a_0 + a_1 + 2k - b)\} + 2(k - b) + 2\left(\frac{3}{2}k^2 - 2k + a_0 + a_1\right) \geqq$$
$$\geqq 2\left(|G| + \frac{3}{2}k^2 - 3k\right)$$

and the result follows in this case.

If $\mathscr{C}(G) = \emptyset$, then $E^k(G) = E(G)$. Hence

$$|E^k(G)| = \frac{1}{2} \sum_{v \in V(G)} d_G(v) \geqq \frac{1}{2} k|G|.$$

On the other hand,

$$\frac{1}{2}k|G| - \left(|G| + \frac{3}{2}k^2 - 3k\right) = |G|\left(\frac{1}{2}k - 1\right) - \frac{3}{2}k^2 + 3k \geqq$$
$$\geqq \frac{1}{2}(k-2)\,3k - \frac{3}{2}k^2 + 3k \geqq 0.$$

Hence the result follows. ∎

We remark that the lower bound of Theorem 8 is sharp. Let l be an integer which is not less than three, $m = \lfloor k/2 \rfloor$, and $n = \lceil k/2 \rceil$. Let $A_1, \ldots, A_l$, $B_1, \ldots, B_l$ be disjoint sets. Assume $|A_i| = m$, say $A_i = \{x_{i,1}, \ldots, x_{i,m}\}$, and $|B_i| = n$, say $B_i = \{y_{i,1}, \ldots, y_{i,n}\}$ $(1 \leqq i \leqq l)$. We define a graph $G(k, l)$ by

$$V(G(k, l)) = \bigcup_{i=1}^{l} (A_i \cup B_i)$$

and

$$E(G(k,l)) = \{x_{h,i}y_{h,j} \mid 1 \leqq h \leqq l,\ 1 \leqq i \leqq m,\ 1 \leqq j \leqq n\}$$
$$\cup \{x_{1,i}x_{2,j},\ x_{l-1,i}x_{l,j} \mid 1 \leqq i, j \leqq m\}$$
$$\cup \{y_{1,i}y_{2,j},\ y_{l-1,i}y_{l,j} \mid 1 \leqq i, j \leqq n\}$$
$$\cup \{x_{h,i}x_{h+1,j} \mid 2 \leqq h \leqq l-2,\ 1 \leqq i \leqq j \leqq m\}$$
$$\cup \{y_{h,i}y_{h+1,j} \mid 2 \leqq h \leqq l-2,\ 1 \leqq i \leqq j \leqq n\}.$$

Then $G(k,l)$ is k-connected, has girth four and

$$E^k(G(k,l)) = \{x_{1,i}y_{1,j},\ x_{l,i}y_{l,j} \mid 1 \leqq i \leqq m,\ 1 \leqq j \leqq n\}$$
$$\cup \{x_{1,i}x_{2,j},\ x_{l-1,i}x_{l,j} \mid 1 \leqq i, j \leqq m\}$$
$$\cup \{y_{1,i}y_{2,j},\ x_{l-1,i}x_{l,j} \mid 1 \leqq i, j \leqq n\}$$
$$\cup \{x_{h,i}x_{h+1,i} \mid 2 \leqq h \leqq l-2,\ 1 \leqq i \leqq m\}$$
$$\cup \{y_{h,i}y_{h+1,i} \mid 2 \leqq h \leqq l-2,\ 1 \leqq i \leqq n\}.$$

Hence

$$|E^{(k)}(G(k,l))| = 2mn+2m^2+2n^2+(l-3)(m+n) = kl-3k+2m^2+2mn+2n^2.$$

If k is even, then $m=n=k/2$ and

$$|E^{(k)}(G(k,l))| = kl-3k+\frac{3}{2}k^2 = |G|+\frac{3}{2}k^2-3k.$$

If k is odd, then $m=(k-1)/2$ and $n=(k+1)/2$. By a simple calculation we have $|E^{(k)}(G(k,l))| = |G|+(3/2)k^2-3k+1/2$. Hence we have infinitely many k-connected triangle-free graphs which attain the lower bound of Theorem 8.

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